

Pattern Selection in the Complex Ginzburg-Landau Equation with Multi-Resonant Forcing

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We study the excitation of spatial patterns by resonant, multi-frequency forcing in systems undergoing a Hopf bifurcation to spatially homogeneous oscillations. Using weakly nonlinear analysis we show that for small amplitudes only stripe or hexagon patterns are linearly stable, whereas square patterns and patterns involving more than three modes are unstable. In the case of hexagon patterns up- and down-hexagons can be simultaneously stable. The third-order, weakly nonlinear analysis predicts stable square patterns and super-hexagons for larger amplitudes. Direct simulations show, however, that in this regime the third-order weakly nonlinear analysis is insufficient, and these patterns are, in fact unstable.

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The variety of patterns and planforms that have been observed in surface waves on vertically vibrated fluid surfaces (Faraday waves) is remarkable [1, 2, 3, 4]. As elucidated in various theoretical investigations [5, 6, 7] this extreme variability is a result of the fact that the wave form of the vibration's forcing function allows for a detailed tuning of various aspects of the interaction between different plane-wave modes, which can stabilize complex patterns like super-lattice patterns and quasi-patterns. Motivated by this richness of patterns we are investigating here the effect of time-periodic forcing with different wave forms on systems undergoing a Hopf bifurcation to spatially homogeneous oscillations.

In order to describe a forced Hopf bifurcation within a weakly nonlinear framework the forcing must be sufficiently weak. For it to have a qualitative effect on the system it must then include frequencies that are close to one or more of the low-order resonances of the system, i.e. its spectrum has to contain frequencies close to the Hopf frequency ω_h itself (1:1-forcing), close to $2\omega_h$ (2:1-forcing), or close to $3\omega_h$ (3:1-forcing). These strong resonances lead to additional terms in the weakly nonlinear description and qualitatively affect the system [8]. To wit, in the weakly nonlinear regime the complex oscillation amplitude A satisfies a complex Ginzburg-Landau equation of the form

$$\begin{aligned} \frac{dA}{dt} = & (\mu + i\nu)A + (1 + i\beta)\Delta A - (1 + i\alpha)A|A|^2 + \\ & + \gamma\bar{A} + \eta\bar{A}^2 + \zeta. \end{aligned} \quad (1)$$

The forcing terms ζ , $\gamma\bar{A}$, and $\eta\bar{A}^2$ express the effect of forcing the system at the frequencies ω , 2ω , and 3ω , respectively, with $\omega = \omega_h + \frac{\mu}{2}$. The parameter μ expresses the distance from the Hopf bifurcation, which is shifted by a $\mathcal{O}(|\eta|^2)$ compared to the unforced case. Here we will focus on the case $\zeta = 0$. To include the forcing near the 1 : 1-resonance one can eliminate the inhomogeneous term ζ by shifting A by the fixed-point solution A_0 satisfying $\zeta = -(\mu + i\nu)A_0 + (1 + i\alpha)A_0|A_0|^2 - \gamma\bar{A}_0 - \eta\bar{A}_0^2$ and use A_0 instead of ζ as an external parameter [9].

As is apparent from (1), the forced Hopf bifurcation is described by an equation that is very similar to a two-component reaction-diffusion equation. The only and significant difference is the term involving β which characterizes the dispersion of unforced traveling wave solutions, which would be absent in the reaction-diffusion context. It plays, however, an essential role in exciting patterns with a characteristic wavenumber [10] and cannot be omitted (cf. (2) below). Pattern selection in a general two-component reaction-diffusion system has been studied in detail by Judd and Silber [11], who find that in principle not only stripe and hexagon patterns can be stable in such systems, but also super-square and super-hexagon patterns. They also find that despite the large number of parameters characterizing these systems surprisingly only few, very special combinations of the parameters enter the equations determining the pattern selection.

Amplitude Equations. In this paper we will stay below the Hopf bifurcation taking $\mu < 0$. Thus, as in Faraday systems, in the absence of forcing, no oscillations arise. To investigate the weakly nonlinear stable standing wave patterns possible in (1) we derive amplitude equations for spatially periodic planforms. The linear stability of the state $A = 0$ is easily obtained by splitting the equation and the amplitude A into real and imaginary parts ($A \equiv A_r + iA_i$). The usual Fourier ansatz $A_{r,i} \propto e^{ikx}$ yields then the neutral stability curve $\gamma_n(k)$ with the basic state being unstable for $\gamma > \gamma_n(k)$. The minimum $\gamma_c(k)$ of the neutral curve is found to be at

$$k_c^2 = \frac{\mu + \nu\beta}{1 + \beta^2}, \quad \gamma_c^2 = \frac{(\nu - \mu\beta)^2}{(1 + \beta^2)}. \quad (2)$$

Since $\mu < 0$, the condition $k_c^2 > 0$ implies that spatial patterns arise only if the detuning of the forcing relative to the Hopf frequency is such that waves with non-zero k are closer to resonance than homogeneous oscillations with $k = 0$ [10]. A typical neutral curve is illustrated in Figure 1 for $\mu = -1$, $\nu = 4$, $\beta = 3$ and $\zeta = 0$. The weakly nonlinear analysis presented in this paper is valid for val-

ues of γ near γ_c . The range of validity is restricted by $\gamma_n(k=0)$ where spatially homogeneous oscillations are excited by the forcing, which interact with the standing-wave modes with wavenumber k_c .

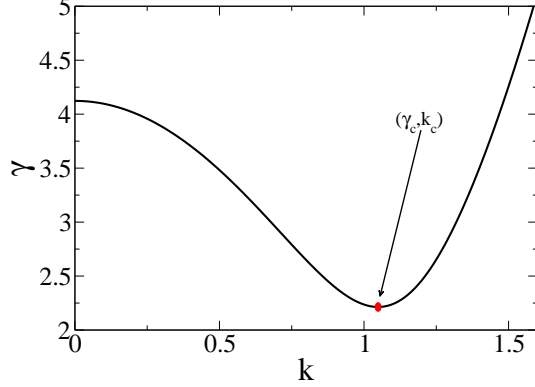


Figure 1: Neutral stability curve for (1) with $\mu = -1$, $\beta = 3$, $\nu = 4$ and $\zeta = 0$. The critical point $(k_c, \gamma_c) = \left(\sqrt{\frac{11}{10}}, \frac{7}{\sqrt{10}}\right)$ is marked by a red circle.

To determine the stability of the various planforms we first determine the amplitude equations for rectangle patterns, which are comprised of two modes separated by an angle θ in Fourier space. We expand (A_r, A_i) as

$$\begin{pmatrix} A_r \\ A_i \end{pmatrix} = \epsilon \left\{ \sum_{j=1, \theta} Z_j(T) e^{i\mathbf{k}_j \cdot \mathbf{r}} + c.c. \right\} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \mathcal{O}(\epsilon^2), \quad (3)$$

where $0 < \epsilon \ll 1$ and the complex amplitudes $Z_1(T)$ and $Z_\theta(T)$ depend on the slow time $T = \epsilon^2 t$. The wavevectors are given by $\mathbf{k}_1 = (k_c, 0)$ and $\mathbf{k}_\theta = (k_c \cos(\theta), k_c \sin(\theta))$. We also expand γ as $\gamma = \gamma_c + \epsilon^2 \gamma_2$.

The usual expansion leads to the amplitude equations for (Z_1, Z_θ) ,

$$\frac{dZ_1}{dT} = \lambda(\gamma - \gamma_c)Z_1 - (b_0|Z_1|^2 + b_1(\theta)|Z_\theta|^2)Z_1, \quad (4)$$

$$\frac{dZ_\theta}{dT} = \lambda(\gamma - \gamma_c)Z_\theta - (b_1(\theta)|Z_1|^2 + b_0|Z_\theta|^2)Z_\theta. \quad (5)$$

If $\theta = \frac{n\pi}{3}$, $n \in \mathbb{Z}$, the quadratic nonlinearity induces a secular term and the expansion has to include three modes rotate by 120° relative to each other,

$$\begin{pmatrix} A_r \\ A_i \end{pmatrix} = \epsilon \left\{ \sum_{j=1}^3 Z_j(T) e^{i\mathbf{k}_j \cdot \mathbf{r}} + c.c. \right\} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \mathcal{O}(\epsilon^2). \quad (6)$$

The parameters can be chosen such that the quadratic solvability condition is delayed to cubic order, yielding

$$\begin{aligned} \frac{dZ_1}{dT} = & \lambda(\gamma - \gamma_c)Z_1 + \sigma \bar{Z}_2 \bar{Z}_3 - \\ & - (b_0|Z_1|^2 + b_2(|Z_2|^2 + |Z_3|^2))Z_1, \end{aligned} \quad (7)$$

with similar equations for Z_2 and Z_3 .

More complex patterns can be described by combining these two analyses. For example, a super-hexagon pattern comprised of two hexagon patterns $\{Z_1, Z_2, Z_3\}$ and $\{Z_4, Z_5, Z_6\}$ that are rotated relative to each other by an angle θ_{SH} is described by the amplitude equation

$$\begin{aligned} \frac{dZ_1}{dT} = & \lambda(\gamma - \gamma_c)Z_1 + \sigma \bar{Z}_2 \bar{Z}_3 - \\ & - (b_0|Z_1|^2 + b_2(|Z_2|^2 + |Z_3|^2))Z_1 \\ & - \sum_{j=0}^2 b_1(\theta_{SH} + j\frac{2\pi}{3})|Z_{4+j}|^2 Z_1 \end{aligned}$$

and corresponding equations for Z_j , $j = 2, \dots, 6$.

The coefficients of the amplitude equations can be written in a simple form, setting $\eta \equiv \eta_r + i\eta_i$:

$$\lambda = \frac{\sqrt{1+\beta^2}}{|\beta|}, \quad \sigma = \frac{2\sqrt{1+\beta^2}(a\eta_r + \eta_i)}{\beta\sqrt{1+a^2}}, \quad (8)$$

$$b_0 = 3\psi + \frac{76}{9}\chi, \quad (9)$$

$$b_1(\theta) = 6\psi + 8f(\theta)\chi, \quad (10)$$

$$b_2 = 6\psi + 10\chi + \phi \quad (11)$$

with

$$\psi = \frac{-2a(\alpha - \beta)\sqrt{1+\beta^2}}{\beta(1+a^2)}, \quad (12)$$

$$\chi = \frac{-\beta(\nu - \mu\beta)}{2(\mu + \nu\beta)}\sigma^2, \quad (13)$$

$$\phi = \frac{4(1+\beta^2)^{\frac{3}{2}}\eta_i[(a+\beta)\eta_r - (a\beta-1)\eta_i]}{\beta^2(1+a^2)(\nu - \mu\beta)} \quad (14)$$

and the angle dependence given by

$$f(\theta) = \frac{3 + 16 \cos^4 \theta}{(4 \cos^2 \theta - 1)^2}.$$

Here $a = \sqrt{1+\beta^2} + \beta$. The scaling of the nonlinear coefficients is based on a normalized eigenvector $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$.

Pattern Selection. As shown by Judd and Silber [11] for general two-component reaction-diffusion systems, at the point of degeneracy at which the quadratic coefficient σ and with it the coefficient χ vanishes not only stripe patterns but also hexagon or triangle patterns can be stable. The conditions for hexagons (or triangles) to be stable are

$$\phi < 0 \quad (15)$$

$$-\frac{2\phi}{15} < \psi < -\frac{\phi}{3}. \quad (16)$$

Whether hexagon or triangle patterns are stable depends on higher-order terms in the amplitude equations [12], which are not considered here. For $0 < \psi < -\frac{2\phi}{15}$,

hexagons are unstable to stripes. Both patterns bifurcate unstably for $\psi < 0$.

Comparing the conditions (15,16) with expressions (12,13,14) shows that in the system considered here stripe patterns and hexagons can be stable at the point of degeneracy, depending on the system parameters α and β . Specifically, at the degeneracy one has $\eta_i = -a\eta_r$, implying

$$\phi(\sigma = 0) = -\frac{4a\eta_r^2(1+\beta^2)^{\frac{3}{2}}}{\beta(\nu - \mu\beta)} < 0. \quad (17)$$

Here we have made use of the fact that $\beta(\nu - \mu\beta) > 0$, which follows from $\mu < 0$ and the condition $k_c^2 > 0$. Whether condition (16) is then satisfied depends on the specific unforced system: only a certain range of values of α leads to stable hexagons, which is determined by the α -dependence of ψ . A distinguishing feature of these hexagon patterns is that both ‘up’- and ‘down’-hexagons are simultaneously stable and are likely to form competing domains. Fig.2 shows an example of the competition between ‘up’- and ‘down’-hexagons in a numerical simulation of (1).

Unfolding the degeneracy, i.e. taking $0 < |\sigma| \ll 1$, the transition to hexagons becomes transcritical and hexagons are stable to stripes for a γ -range given by

$$\gamma_c - \frac{\sigma^2}{4\lambda(b_0 + 2b_2)} < \gamma < \gamma_c + \frac{\sigma^2(2b_0 + b_2)}{\lambda(b_2 - b_0)^2} \equiv \gamma_{HS}.$$

Note that $\gamma_{HS} > \gamma_c$, even if (15,16) are not satisfied, since stripes do not exist for $\gamma < \gamma_c$. The instability of hexagons at γ_{HS} only arises if $b_2 > b_0$, that is, if $3\psi + 14\chi/9 + \phi > 0$. With $\sigma \neq 0$ the up-down symmetry of the amplitude equations (7) is broken and, depending on the sign of σ , either up- or down-hexagons are preferred.

Turning to other planforms, Judd and Silber found that rectangular planforms cannot be stable at or near the degeneracy point [11]. Interestingly, however, they find that while super-hexagons cannot be stable at the degeneracy point, they can arise in a very small parameter regime in its vicinity if the conditions

$$\phi > 0, \quad (18)$$

$$-\frac{\phi}{21} < \psi < \frac{\phi}{3} \quad (19)$$

are met. They then can be bistable with hexagons. We find that in our system the condition (18) cannot be met. Rewriting ϕ in terms of the deviation σ from the degeneracy condition,

$$\phi = -\frac{2(1+\beta^2)}{a\beta(\nu - \mu\beta)} \left(2\sqrt{1+\beta^2}\eta_i^2 - \frac{(a+\beta)\eta_i}{\sqrt{1+a^2}}\sigma \right),$$

shows that - for small $|\sigma|$ - ϕ can be made positive only by making η_i small as well ($\eta_i = \mathcal{O}(\sigma)$). Even then ϕ can only be slightly positive, $\phi = \mathcal{O}(\sigma)$, requiring that $\psi = \mathcal{O}(\sigma)$ in order to satisfy (19). Under these conditions

all cubic coefficients in (9,10,11) would become of $\mathcal{O}(\sigma)$ and without knowledge of the next-order coefficients no stability predictions can be made.

Away from the degeneracy, for $\sigma = \mathcal{O}(1)$, the above arguments suggest that it should be possible to satisfy the stability conditions (18,19). For $\sigma = \mathcal{O}(1)$ they are, however, not the correct stability conditions since they ignore the angle dependence of the cubic coefficients, which is $\mathcal{O}(\sigma)$. We use (18,19) therefore only as a guide to locate parameter regimes in which super-hexagons may be expected to be stable and then determine the full stability eigenvalues. One such case is given by the linear parameters used in Figure 1 with the nonlinear parameters $\alpha = -1$ and $\eta = e^{i\pi/4}$.

Moreover, for $\sigma = \mathcal{O}(1)$ the weakly nonlinear analysis predicts also stable rectangle patterns satisfying $|b_1(\theta)| < b_0$ if

$$\alpha > |\beta| \left[1 - \frac{1+a^2}{6a\sqrt{1+\beta^2}} \left(\frac{76}{9} - f(\theta) \right) \chi \right].$$

Within the hexagon sub-space, for $|b_2/b_0| > 1$ hexagons become unstable to stripes as γ is increased beyond γ_{HS} . Satisfied simultaneously, the two conditions $|b_2/b_0| > 1$ and $|b_1(\theta)| < b_0$ would yield a situation in which hexagons are unstable to stripes, which in turn are unstable to rectangle patterns.

Using direct numerical simulations of the forced complex Ginzburg-Landau equation (1) we have studied to what extent the predictions of the weakly nonlinear analysis are borne out. In the degenerate case $\sigma = 0$ we find, as predicted, either stripes or hexagons to be stable depending on the value of α chosen. A typical hexagonal pattern obtained from random initial condition is shown in Fig.2. As expected, it exhibits competing domains of up- and down-hexagons.

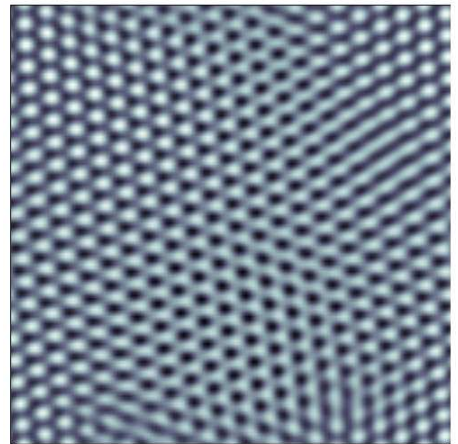


Figure 2: Competing domains of up- and down-hexagon domains starting from random initial conditions with linear parameters as in Fig. 1 and nonlinear parameters $\alpha = -1$, $\eta_r = 0.4$ and $\eta_i = 0.4(\sqrt{10} + 3)$. η_r and η_i are chosen so that $\sigma = 0$.

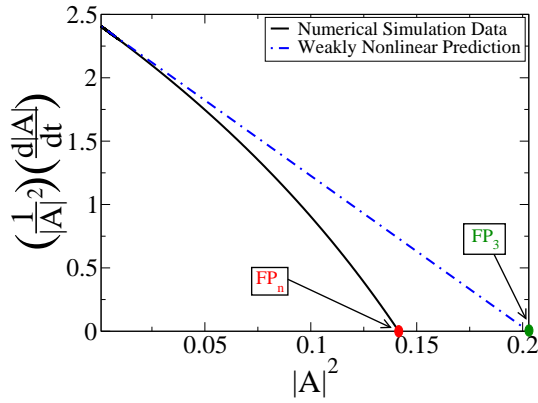


Figure 3: Comparing amplitude from numerical simulation with amplitude predicted by weakly nonlinear analysis. Simulation parameters: $\mu = -1, \nu = 4, \beta = 3, \alpha = -1, \eta = e^{i\pi/4}$. FP_n corresponds to the fixed point obtained from numerical simulation, FP_3 to that obtained from the weakly nonlinear calculation to cubic order.

Away from the degeneracy, $\sigma = \mathcal{O}(1)$, the validity of the weakly nonlinear analysis can be severely restricted by the fact that the amplitudes of all stable branches are $\mathcal{O}(1)$, which formally suggests the significance of higher-order terms in the expansion. Indeed, in and near the parameter regimes for which the weakly nonlinear analysis predicts stable super-hexagon patterns we do not find any indication of their stability. To assess explicitly the significance of the higher-order terms in the amplitude equations for $\sigma = \mathcal{O}(1)$ we extract them directly from the numerical simulation for the case of hexagon patterns. Fig. 3 shows the numerically determined dependence of $|A|^{-2} d|A|/dt$ on $|A|$ for $\gamma = \gamma_c$. For very small $|A|$ it agrees well with the result obtained by the third-order weakly nonlinear theory, which yields the straight dashed line. However, even for $\gamma = \gamma_c$ the amplitude $|A|$ saturates only at a value of $|A| \approx 0.14$ for which the third-order theory deviates significantly from the full result. A fit of $|A|^{-2} d|A|/dt$ to a higher-order polynomial shows that the magnitude of the quintic term in the amplitude equation reaches a value of 20% of the cubic term. The fixed point obtained from weakly nonlinear analysis to cubic truncation deviates from the numerically obtained fixed point by 30%. This supports our interpretation that in this regime the cubic amplitude equation does

not allow quantitative predictions.

In summary, we have investigated the regular spatial planforms that can be stably excited in a system undergoing a Hopf bifurcation by applying a periodic forcing function that resonates with the second and third harmonic of the Hopf frequency. We have done so within the weakly nonlinear regime by deriving the appropriate amplitude equations describing the selection between various planforms. By tuning the phase of the forcing close to $3\omega_h$ one can always reach the point of degeneracy where no quadratic terms arise in the amplitude equations, despite the quadratic interaction in the underlying complex Ginzburg-Landau equation. Hexagons arise then in a supercritical pitchfork bifurcation. Depending on the parameters of the unforced system we find that in this regime either the hexagon patterns or stripe patterns can be stable. In the former case competing domains of up- and down-hexagons are found in numerical simulations when starting from random initial conditions. Surprisingly, despite the extensive control afforded by the two forcing terms, no square, rectangle, or super-hexagon patterns are stable in the vicinity of this degeneracy, irrespective of the parameters of the unforced system. Only in the regime in which hexagons arise in a strongly transcritical bifurcation the weakly nonlinear theory predicts the possibility of stable rectangles or super-hexagons. There, however, direct numerical simulations of the complex Ginzburg-Landau equation indicate no such stability and we show that terms of higher order in the amplitudes are relevant.

By introducing a further forcing frequency, which is also close to the 2:1-resonance, the transcritical bifurcation to hexagons can be avoided. As we show in a separate publication, the corresponding, more elaborate weakly nonlinear theory then correctly predicts stable quasi-patterns comprised of four, five, or more modes [13].

For larger values of the forcing the spatially periodic standing wave modes interact with a spatially homogeneous oscillation that is also excited by the forcing. The interaction between these two types of modes could lead to interesting patterns, which are, however, beyond the scope of this paper.

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